$$w\bigg|_{f=1} = \frac{dw}{df}\bigg|_{f=1} = \left[\frac{d^2\phi}{df^2} - \frac{\mu}{f}\frac{d\phi}{df}\right]_{f=1} = 0 \tag{29}$$

The last boundary condition representing the vanishing of the circumferential strain was proposed by Reissner[8] and Gradowczyk.[9]

It is interesting to note that if we put $a = b$, and $K_x = K_y = K$ the solution coincides exactly with that obtained by Gradowczyk[9] for a clamped spherical dome with curvature $K$. Further in the limiting case as $K_x$ and $K_y$ tend to zero we obtain

$$\lim_{K_x, K_y \to 0} W = q_1(1-f^2)^2/4 = q_1(1-x^2/a^2 - y^2/b^2)^2/4 \tag{30}$$

which is the exact expression for the bending of a clamped, uniformly loaded elliptic plate.

### References


1 Mazumdar, J., "A Method for Solving Problems of Elastic Plates of Arbitrary Shape," *Journal of Australian Mathematical Society*, Vol. 11, 1970, pp. 95–112.

2 Mazumdar, J., "Transverse Vibration of Elastic Plates by the Method of Constant Deflection Lines," *Journal of Sound and Vibration*, Vol. 18, 1971, pp. 147–155.

3 Mazumdar, J., "Buckling of Elastic Plates by the Method of Constant Deflection Lines," *Journal of Australian Mathematical Society*, Vol. 13, 1972, pp. 91–103.

4 Mazumdar, J., "Transverse Vibration of Membranes of Arbitrary Shape by the Method of Constant Deflection Contours," *Journal of Sound and Vibration*, Vol. 27, 1973, pp. 47–57.

5 Mazumdar, J. and Jones, R., "A Simplified Approach to the Analysis of Large Deflections of Plates," *Journal of Applied Mechanics* (in press).

6 Jones, R., "Application of the Method of Constant Deflection Contour Lines to Elastic Plates and Shell Problems," Ph.D. thesis, 1973, The University of Adelaide, Adelaide, South Australia.

7 Timoshenko, S. P. and Woinowsky-Krieger, S., *Theory of Plates and Shells*, McGraw-Hill, New York, 1959.

8 Reissner, E., "Stresses and Small Displacements of Shallow Spherical Shells," *Journal of Mathematics and Physics*, Vol. 25, 1946, pp. 80–85, 279–300.

9 Gradowczyk, M. H., "Some Remarks on the Theory of Shallow Spherical Shells," *Ingenieur Archiv.*, Vol. 32, 1963, pp. 297–303.


# Note on Boundary Layer in a Dusty Gas


K. M. Chakrabarti*

*Regional Engineering College, Durgapur, West Bengal, India*


### Introduction

SPROULL[1] has observed that adding dust to air flowing in turbulent motion through a pipe appreciably reduces the resistance coefficient. Saffman[2] discussed in detail the stability of laminar flow of a dusty gas. Michael[3,4] has investigated the plane parallel flow of a dusty gas. It is supposed that the dust particles are uniform in size and shape, so that the stability of the flow is conserved. This Note initiates boundary-layer theory for a dusty gas past an infinite plate in the primary stages of the motion (Rayleigh's Case). We have discussed the effect on the boundary layer of the potential flow of a gas in presence of dust particles. The density of the dust material is allowed to be large compared with the gas density, so that the mass concentration of the dust $f$ is small. The expression for local skin friction due to the dusty gas has been deduced and has been compared with that for the case of a clean gas. It is seen that the effect of dust reduces the local skin friction. This reduction tends to diminish as $t$ increases. However, this analysis is valid only at the initial stage of a boundary-layer development, as in the case of Lord Rayleigh.

### Formulation of Equations and Solutions

Let the velocity and number density of the dust particles be described by the fields $\mathbf{v}(\mathbf{x}, t)$ and $N(\mathbf{x}, t)$. Assuming that the particles of the dust are small enough to make the Stokes law of resistance between the dust particles and gas appropriate, the equations of motion become

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\operatorname{grad} p + \nu\nabla^2\mathbf{u} + \frac{kN}{\rho}(\mathbf{v}-\mathbf{u}) \tag{1}$$

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = k(\mathbf{u}-\mathbf{v}) \tag{2}$$

$$\operatorname{div}\mathbf{u} = 0 \tag{3}$$

$$\frac{\partial N}{\partial t} + \operatorname{div} N\mathbf{v} = 0 \tag{4}$$

In these equations, $\mathbf{u}$ and $\mathbf{v}$ are the velocities of the gas and dust particles, respectively, $k$ being the Stokes resistance coefficient which, for spherical particles of radius $a$, is $6\pi\mu a$, $\mu$ the viscosity of the gas, $\rho$ the density of the gas, $p$ the gas pressure, and $\tau = m/k$ the relaxation time of the dust particles where $m$ is the mass of the dust particles.

In the present case, we have assumed that the dust is uniformly distributed in the gas and the motion is induced by the potential flow $u_\infty(x, t) = U(x)$. Let $\mathbf{u} = u(y, t)\mathbf{x}$, $\mathbf{v} = v(y, t)\mathbf{x}$, and Eq. (4) is satisfied with $N = N_o$ (a constant) throughout the motion. Equation (3) is identically satisfied. And it remains to solve Eqs. (1) and (2), which can be written as

$$\partial u/\partial t = \nu(\partial^2 u/\partial y^2) + (kN_o/\rho)(v-u) \tag{5}$$

$$\tau(\partial v/\partial t) = u - v \tag{6}$$

We shall work in terms of the dimensionless time variable $\bar{t} = t/\tau$ and the dimensionless length $\bar{y} = y/(\nu\tau)^{1/2}$, in which case the equations become (dropping bars)

$$\partial u/\partial t = (\partial^2 u/\partial y^2) + f(v-u) \tag{7}$$

$$\partial v/\partial t = u - v \tag{8}$$

where $f = N_o m/\rho$ is the mass concentration of dust.

Let us express $u$ and $v$ as follows:

$$u = u_o + f u_1 + f^2 u_2 + \cdots \tag{9}$$

$$v = v_o + f v_1 + f^2 v_2 + \cdots \tag{10}$$

The first and second approximations of relations (9) and (10) are taken, respectively, as $u = u_o$, $v = v_o$ (first approximation); $u = u_o + f u_1$, $v = v_o + f v_1$ (second approximation). Taking the first approximation, Eqs. (7) and (8) become

$$\partial u_o/\partial t = \partial^2 u_o/\partial y^2 \tag{11}$$

$$\partial v_o/\partial t = u_o - v_o \tag{12}$$

with the boundary conditions

$$y = 0,\ u_o = 0; \quad y = \infty,\ u_o = U(x)$$

and initial conditions $t = 0$, $u_o = 0$, $v_o = 0$.

We shall now try to solve Eqs. (11) and (12) with the help of the abovementioned boundary and initial conditions. The solution of Eq. (11) is

$$u_o(x, y, t) = U(x)\cdot\frac{2}{(\pi)^{1/2}}\int_o^\eta e^{-\eta^2} d\eta \tag{13}$$

Where

$$\eta = y/2(t)^{1/2}$$

with the help of Eq. (13), Eq. (12) becomes

$$\frac{\partial v_o}{\partial t} + v_o = \frac{2}{(\pi)^{1/2}} U(x)\int_o^\eta e^{-\eta^2} d\eta \tag{14}$$


Received August 9, 1973; revision received December 12, 1973. The author is very grateful to S. K. Ghoshal of Jadavpur University for his guidance and helpful suggestions in preparing this Note.

Index categories: Boundary Layers and Convective Heat Transfer—Laminar; Nonsteady Aerodynamics.


* Lecturer in Mathematics.

Solution of Eq. (14) is

$$v_0 = \frac{2}{(\pi)^{1/2}} U(x) \left\{ \int_o^t e^t \left[ \int_o^\eta e^{-\eta^2} d\eta \right] dt \right\} e^{-t} \tag{15}$$

which satisfies the initial condition $v_o = 0$ for $t = 0$. It is seen that, as $\eta \to \infty$, we obtain

$$v_o = U(x)[1 - e^{-t}] \tag{16}$$

This shows how the velocity of the dust particles differs with the potential velocity $U(x)$ of the fluid in the mainstream. With the passage of time this difference diminishes and as the motion stabilizes, $v_o \to U(x)$ as $t \to \infty$.

Taking the second approximation, Eqs. (7) and (8) become

$$\partial u_1/\partial t = (\partial^2 u_1/\partial y^2) + (v_o - u_o) \tag{17}$$

and

$$(\partial v_1/\partial t) + v_1 = u_1 \tag{18}$$

By similarity transformation $\eta = y/2(t)^{1/2}$ Eq. (17) becomes

$$(d^2 u_1/d\eta^2) + 2\eta(du_1/d\eta) = F(x, \eta, t) \tag{19}$$

where

$$F(x, \eta, t) = 4t(u_o - v_o) = \frac{8}{(\pi)^{1/2}} U(x) t e^{-t} \int_o^\eta e^{-\eta^2} d\eta$$

Now solution of Eq. (19) is

$$u_1 = \int_o^\eta e^{-\eta^2} \left\{ \int e^{\eta^2} F(x, \eta, t) d\eta \right\} d\eta + C_1 \int_o^\eta e^{-\eta^2} d\eta + C_2$$

when

$$\eta = 0, \ u_1 = 0 \quad \therefore C_2 = 0$$

Since at $\eta = \infty$, $u_1 = 0$, then $C_1$ is given by the relation

$$C_1 = -\frac{16}{\pi} U(x) t e^{-t} \int_o^\infty e^{-\eta^2} \left\{ \int e^{\eta^2} \left( \int_o^\eta e^{-\eta^2} d\eta \right) d\eta \right\} d\eta \tag{20}$$

$$\therefore u_1 = \int_o^\eta e^{-\eta^2} \left\{ \int e^{\eta^2} F(x, \eta, t) d\eta \right\} d\eta + C_1 \int_o^\eta e^{-\eta^2} d\eta \tag{21}$$

where $C_1$ is given by relation (20).

Solution of Eq. (18) is given by

$$v_1 = e^{-t} \int_o^t u_1 e^t dt \tag{22}$$

Modifications of the velocity profiles of $u$ and $v$ are given by Eqs. (21) and (22), respectively.

### Local Skin Friction

If $\tau_D(x)$ is the local skin friction for a dusty gas, then

$$\tau_D(x) = \mu(\partial u/\partial y)_{y=0} = \frac{\mu U(x)}{(\pi t)^{1/2}} \left[ 1 - \frac{8 f t e^{-t}}{(\pi)^{1/2}} \int_o^\infty e^{-\eta^2} \left\{ \int e^{\eta^2} \left( \int_o^\eta e^{-\eta^2} d\eta \right) d\eta \right\} d\eta \right] \tag{23}$$

When $f$ is small, the effect of dust particles is to increase the effective density of the gas. The viscosity remains nearly unaffected, though the kinematic viscosity is reduced by the factor $1/(1+f)$. These increases in density and the reduction of kinematic viscosity results are credited to P. G. Saffman.[2] If $\tau_o(x)$ be the local skin friction for the corresponding clean gas then

$$\tau_o(x) = \mu U(x)/(\pi t)^{1/2} \tag{24}$$

$$\therefore \frac{\tau_D(x)}{\tau_o(x)} = \left[ 1 - \frac{8f}{(\pi)^{1/2}} t e^{-t} \int_0^\infty e^{-\eta^2} \left\{ \int e^{\eta^2} \left( \int_0^\eta e^{-\eta^2} d\eta \right) d\eta \right\} d\eta \right] \tag{25}$$

From Eq. (25) it is seen that if $f$ is nonzero the ratio $[\tau_D(x)/\tau_o(x)] < 1$. Thus, the dust particles reduce the local skin friction. This reduction diminishes as $f$ diminishes and as $t$ increases $\tau_D(x) \to \tau_o(x)$. However, all these analyses are valid (as in Rayleigh's case) in the initial development of the motion.

### References


[1] Sproull, W. T., "Viscosity of Dusty Gases," *Nature*, Vol. 190, 1961, pp. 190, 976.

[2] Saffman, P. G., "On the Stability of Laminar Flow of a Dusty Gas," *Journal of Fluid Mechanics*, Vol. 13, Pt. 1, 1962, pp. 120–128.

[3] Michael, D. H., "Plane Parallel Flow of a Dusty Gas," *Mathematika*, Vol. 13, Pt. 1, 1966, pp. 97–109.

[4] Michael, D. H., "The Steady Motion of a Sphere in a Dusty Gas," *Journal of Fluid Mechanics*, Vol. 31, Pt. 1, 1968, pp. 175–192.


# Influence of Viscosity on the Stability of a Cylindrical Jet


J. N. Anno*

*University of Cincinnati, Cincinnati, Ohio*


In a previous Note,[1] Rayleigh's famous result[2] for the conditions of instability of an inviscid cylindrical jet was derived by considering an integral form of the conservation of mechanical energy. Given in this present Note is an extension of the integral method to an analysis of the conditions for instability of a viscous cylindrical jet. The result obtained is identical to that obtained by Weber[3] in 1931, but the derivation is believed to be more general. Specifically, it is not necessary to assume a jet at rest, nor to explicitly invoke the Navier-Stokes equations.

For a viscous fluid, an energy balance for any closed, stationary control volume of volume $V$ and surface area $S$ can be written as

$$\int_S \tau_{ij} n_j u_i \, dS = \int_S \frac{1}{2} \rho u_j n_j u_i u_i \, dS + \int_V \frac{\partial}{\partial t} \left( \frac{1}{2} \rho u_i u_i \right) dV + \int_V R \, dV \tag{1}$$

where $R$ is the viscous energy dissipation, given by

$$R = (\mu/2)(u_{i,j} + u_{j,i})(u_{i,j} + u_{j,i}) \tag{2}$$

In this notation $\tau_{ij}$ is the stress tensor, $u_i$ is the velocity component in the $i$th direction, $n_j$ is the projection of the outward unit normal to the surface in the $j$th direction, $\rho$ is the fluid density, $\mu$ is the fluid viscosity and, as conventional in tensor notation, repeated indices indicate summation over all coordinates. A comma within a subscript denotes differentiation with respect to the coordinate following it.

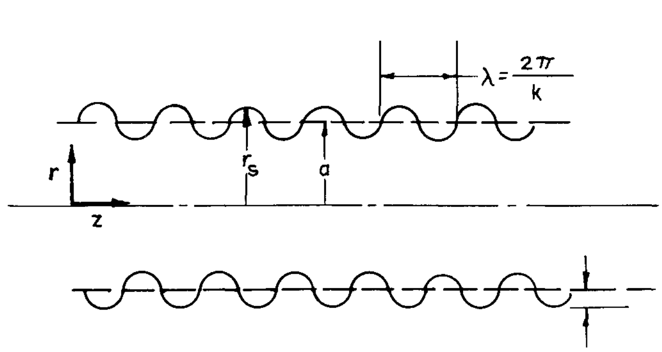


**Fig. 1 Geometry of the cylindrical jet.**


Received September 27, 1973; revision received March 18, 1974.

Index category: Hydrodynamics.



* Professor of Nuclear Engineering, Department of Chemical and Nuclear Engineering.